PHYSICAL REVIEW E **81**, 016606 (2010)

Cloaking of the momentum in acoustic waves

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(Received 11 August 2009; revised manuscript received 28 November 2009; published 14 January 2010)

Through an appropriate change in variables, we find that the three-dimensional acoustic wave equation is subject to the transformation media interpretation. In particular, we determine that this interpretation can be extended beyond the pressure difference to also account for the momentum transported by the wave. The suitability of momentum transport is especially interesting as it is an example where the field of interest is not governed by a wave equation. We examine how both fields behave in the case of cloaking. Explicit consideration of the boundary conditions shows that perfect cloaking is preserved, even when the incoming momentum is nonzero at the surface of the cloak.

DOI: 10.1103/PhysRevE.81.016606 PACS number(s): 43.20.+g

The concept of "transformation media" [1,2] offers an approach to the control of physical phenomena. It has already proven its usefulness with the control of electromagnetic waves [3,4], in particular the creation of a cloak of invisibility [1,3,5]—an application which has been experimentally verified on numerous occasions [5–10]. Similar methods have been developed for the manipulation of electromagnetic potentials through control of conductivity [11,12] as well as matter waves of specified energy through control of effective mass and potential [13]. The development of transformation media theory for sound waves bears an especially dramatic history. Milton *et al.* [14] examined the most general form of the equations, but did not succeed in determining the proper transformations for the generic case. Cummer and Schurig [15] specialized to the experimentally important case of linearized waves in inviscid flow. This approach gave a pair of coupled, linear partial differential equations, such as are typically found in the derivation of wave equations. They then specialized further and restricted attention to two-dimensional flows, which allowed for an analogy with Maxwell's equations. Chen and Chan [16] started with the wave equation for pressure alone and used an analogy with the electromagnetic potential to determine the three-dimensional (3D) approach. A similar approach, adopted independently by Cummer *et al.* [17], was published shortly after Chen and Chan's work and also demonstrates 3D cloaking.

The work of Chen and Chan, then, would seem to have settled the development of transformation media theory for acoustic waves—especially in light of the experimental confirmation of cloaking in metafluids [18,19]. However, their approach to acoustic waves neglected to account for the second dynamical variable in fluid mechanics—they demonstrated the possibility of cloaking for pressure but not for velocity. Since there is no reason, *a priori*, to prefer the transformation of the pressure wave to that of the velocity wave, it is worthwhile to see if the velocity wave also transforms properly. Cummer *et al.* did consider the velocity as well as the pressure, but they only demonstrated the proper transformation of pressure. They relied upon the coupling of the set of partial differential equations to determine the velocity from the properly transformed pressure. After that, they demonstrated that handling the velocity in this manner will satisfy the requirements for cloaking. However, the coupling of the set of partial differential equations does not establish the proper transformation of the velocity, as a consideration of Maxwell's equations will indicate. Faraday's Law states that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{1}$$

implying that the two fields are coupled. But while the $\vec{E}$ field is subject to the transformation media interpretation, it is the $\vec{H}$ field and not the $\vec{B}$ field that also transforms properly.

As a start to the derivation, we consider the coupled partial differential equations used by Cummer and Schurig [15],

$$\rho\frac{\partial}{\partial t}\vec{v} = -\vec{\nabla}p, \tag{2}$$

$$\frac{\partial}{\partial t}p = -\lambda\vec{\nabla}\cdot\vec{v}. \tag{3}$$

Applying the $\frac{\partial}{\partial t}$ operator to the second equation and using the commutativity of mixed partials will recover the wave equation considered by Chen and Chan [16]. Instead, we specialize to stationary ρ and λ and apply $\frac{\partial}{\partial t}$ to the first equation. This gives

$$\rho_{ij}\partial_t^2 v^j = \partial_i(\lambda\partial_j v^j) \tag{4}$$

where we have used Einstein's summation convention in Cartesian coordinates. Notice that this equation is not the wave equation (even for constant ρ and λ), since the operator on the right-hand side is not the Laplacian [20]. To determine the proper structure of the transformation medium for velocity, it is necessary to rewrite the above equation in general coordinates (with isotropic material parameters),

$$\rho\partial_t^2 v_i = \partial_i\left[\frac{\lambda}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}v_k)\right]. \tag{5}$$

Comparing Eqs. (4) and (5) gives the following rules for the transformation medium:

$$\bar{\lambda} = \frac{\lambda}{\sqrt{g}}, \tag{6}$$

$$\bar{v}^i = \sqrt{g} g^{ij} v_j, \tag{7}$$

$$\bar{\rho}_{ij} = \frac{\rho}{\sqrt{g}} g_{ij}. \tag{8}$$

Clearly, then, the velocity wave does not follow the transformation media interpretation. However, this does not contradict the positive result of pressure waves experiencing effective media. The dilemma is very quickly resolved by considering a proper choice of variables. In this solution, the result is very similar to that of electromagnetic fields in matter, although explicit analogies between the acoustic and electromagnetic fields are not required. Considering the equation $\partial_t(\rho\vec{v}) = -\vec{\nabla} p$ and the fact that the gradient operator is independent of choice of coordinates, it is clear that the proper choice of variables is $\rho\vec{v}$ and p, not $\vec{v}$ and p. For notational convenience, we denote

$$\vec{P} \equiv \rho\vec{v}. \tag{9}$$

The variable $\vec{P}$ has two possible interpretations. It can be seen as the momentum density of the wave, which is the most obvious approach. However, it has dimensions $\text{kg}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$, implying that it can also be interpreted as the mass flux per unit area. Since it is more succinct, we shall refer to $\vec{P}$ as the momentum for the remainder of the paper.

The resultant set of differential equations is

$$\frac{\partial}{\partial t}\vec{P} = -\vec{\nabla} p, \tag{10}$$

$$\frac{\partial}{\partial t} p = -\lambda \vec{\nabla} \cdot (\rho^{-1}\vec{P}). \tag{11}$$

Once again, applying the derivative operator $\frac{\partial}{\partial t}$ to the second equation will reproduce the pressure wave equation, the pressure cannot change in this change of variables. On the other hand, the equation of motion for the second dynamical variable is

$$\frac{\partial^2}{\partial t^2}\vec{P} = \vec{\nabla}[\lambda \vec{\nabla} \cdot (\rho^{-1}\vec{P})]. \tag{12}$$

In general coordinates with isotropic material parameters, Eq. (12) becomes

$$\partial_t^2 P_i = \partial_i \left[\frac{\lambda}{\sqrt{g}} \partial_j (\sqrt{g} g^{jk} \rho^{-1} P_k) \right]. \tag{13}$$

And finally, in Cartesian coordinates and anisotropic density, Eq. (12) is

$$\partial_t^2 \bar{P}_i = \partial_i [\bar{\lambda} \partial_j (\overline{\rho_{jk}^{-1}} \bar{P}_k)]. \tag{14}$$

The transformation for the variables is therefore

$$\bar{\lambda} = \frac{\lambda}{\sqrt{g}}, \tag{15}$$

$$\overline{\rho_{ij}^{-1}} = \sqrt{g} g^{ij} \rho^{-1}, \tag{16}$$

$$\bar{\rho}_{ij} = \frac{\rho}{\sqrt{g}} g_{ij}, \tag{17}$$

$$\bar{P}_i = P_i, \tag{18}$$

which is exactly what is required to support the transformation medium interpretation, as desired. What is more, the transformations for λ and ρ are same for both p and $\vec{P}$ [15–17]. This implies that both dynamical variables can be appropriately transformed by the same change of material parameters, as we would desire for the proper functioning of a device. (This congruence is not necessarily a corollary of each field separately transforming properly or of the fields being coupled; it is not beyond the realm of possibility for there to be overspecified material transformations.)

It is interesting to note that this is an example of a transformation media field that is not subject to the wave equation. However, the equation for $\vec{P}$ is more complicated and less familiar than p's wave equation, so it is a significant mathematical simplification to treat p in isolation. It is then sufficient to rely upon the proper transformation of $\vec{P}$ and its coupling to p through the original set of partial differential equations [Eqs. (10) and (11)] to determine all the dynamical variables. This trade off is rather similar to the preference for the potential V in electrostatics to the field $\vec{E}$—it is considerably easier to work with the scalar field. In addition, pressure and momentum are on equal footing in acoustics. So unless there is a specific reason to turn to the momentum equation, it is unlikely to be worth consideration.

The other side of this nicety in the different behaviors of $\vec{P}$ and $\vec{v}$ is to wonder if there are practical differences in the behavior of the two fields. In particular, is it possible to determine the presence of a cloak by studying the velocity of the wave? In an arbitrary location, the answer is clearly: no. Partial differential Eqs. (2) and (3) serve to couple the velocity to the pressure everywhere. So as long as everything remains analytic (and it is possible to take derivatives), then the velocity behaves properly. It is worth particular attention, however, when the other variables become singular. When expressions such as $\vec{v} = \rho^{-1}\vec{P}$ or $\vec{\nabla}\cdot\vec{v} = -\lambda^{-1}\partial_t p$ involve indeterminate forms, then things are far less simple. Nor is the question purely academic, as this is exactly what occurs at the inner lining of the cloak—presenting the possibility that objects inside of a cloak could still partially hear the outside noise (or, by analogy with electromagnetism, still partially see out by measuring the $\vec{B}$ field).

To investigate this possibility, we adopt the design presented in Chen and Chan [16] and Cummer *et al.* [17], following Pendry *et al.* [1], and Greenleaf *et al.* [11,12]: we apply a linear transformation to the radial distance $r' = a + (b-a)r/b$, where a is the inner lining of the cloak and b is the outer lining. The angular density becomes

$$\rho'_{\theta\theta} = \frac{b-a}{b}\rho_0, \tag{19}$$

$$\rho'_{\phi\phi} = \rho'_{\theta\theta} \tag{20}$$

and so is everywhere finite. The radial density, however, is

$$\rho'_{rr} = \frac{b-a}{a}\frac{r'^2}{(r'-a)^2}\rho_0 \tag{21}$$

and the bulk modulus is

$$\lambda = \frac{(b-a)^3}{b^3}\frac{r'^2}{(r'-a)^2}\lambda_0. \tag{22}$$

Both of these parameters are therefore singular at $r'=a$ (i.e., at $r=0$). The pressure inside the cloak as a response to an arbitrary input is [16]

$$p = \sum_{lm} p_{lm} \cdot j_l(\omega c_{s0} r) \cdot Y_l^m(\theta,\phi) \tag{23}$$

where Y_l^m is the spherical harmonic, j_l the spherical Bessel function, $r=b(r'-a)/(b-a)$, and c_{s0} is the speed of sound in the isotropic background medium $\sqrt{\rho_0/\lambda_0}$. The summation runs through all integer $l \geq 0$ and all integer m in the range $-l$ to l. For notational convenience, we define $\omega c_{s0} b(r'-a)/(b-a) \equiv kR$ with $R=r'-a$. We now determine $\vec{P}$,

$$P_r = -\frac{k}{i\omega}\sum_{lm} p_{lm}\frac{[l j_{l-1}(kR)-(l+1)j_{l+1}(kR)]}{2l+1}Y_l^m(\theta,\phi), \tag{24}$$

$$P_\theta = -\frac{1}{i\omega r' \sin\theta}\sum_{lm} p_{lm} j_l(kR)[l\cos\theta Y_l^m(\theta,\phi) - (l+m)Y_{l-1}^m(\theta,\phi)], \tag{25}$$

$$P_\phi = -\frac{1}{i\omega r' \sin\theta}\sum_{lm} imp_{lm} j_l(kR) Y_l^m(\theta,\phi), \tag{26}$$

where we have used the recurrence relations for the derivatives of j_l and Y_l^m [21]. Since P_θ and P_ϕ are finite (0 at the inner lining), they are everywhere determinate. We can ignore them in the further derivation, as their behavior is trivially coupled to p. At $kR=0$,

$$P_r = -\frac{k}{3i\omega}\sum_m p_{1m}Y_1^m(\theta,\phi) - \frac{k}{i\omega}\sum_{l\neq 1,m} p_{lm}\frac{[l j_{l-1}(0)-(l+1)j_{l+1}(0)]}{(2l+1)}Y_l^m(\theta,\phi) \tag{27}$$

and $j_{l\pm1}(0)=0$ for $l \neq 1$. So only the $l=1$ components ($m=-1,0,1$) are nonzero. To determine if a nonzero signal is transmitted inside the cloak as a result of the $l=1$ components, it is necessary to fit boundary conditions. The boundary conditions for this system of differential equations are [22,23]

$$\rho_{(a)}^{-1}\vec{P}_{(a)} = \rho_{(b)}^{-1}\vec{P}_{(b)}, \tag{28}$$

$$\lambda_{(a)}\vec{\nabla}\cdot(\rho_{(a)}^{-1}\vec{P}_{(a)}) = \lambda_{(b)}\vec{\nabla}\cdot(\rho_{(b)}^{-1}\vec{P}_{(b)}), \tag{29}$$

$$p_{(a)} = p_{(b)}, \tag{30}$$

$$\rho_{(a)}^{-1}\vec{\nabla}p_{(a)} = \rho_{(b)}^{-1}\vec{\nabla}p_{(b)}. \tag{31}$$

As expected, Eq. (28) implies that only nonzero momenta are transmitted through any barrier. But Eq. (21) at $r'\to a$ implies that $\rho_{(a)}^{-1}\to 0$, so even a nonzero field will not produce a measurable signal within the cloak. The momentum is therefore discontinuous at the inner lining of the cloak. To see this more directly, recall that $v_r=\rho_{rr}^{-1}P_r$, so Eq. (28) refers to the velocity:

$$v_r = -\frac{b}{b-a}\frac{R^2}{r'^2}\frac{k}{i\omega\rho_0}\sum_{lm} p_{lm} \times \frac{[l j_{l-1}(kR)-(l+1)j_{l+1}(kR)]}{2l+1}Y_l^m(\theta,\phi) \tag{32}$$

the $R^2\to 0$ term will ensure that the velocity vanishes everywhere (no indeterminacy). So for both momentum and velocity, perfect cloaking is preserved. This can be easily understood by realizing that the density at the inner lining is infinite, and so no finite amount of pressure will cause it to move (possess a nonzero velocity). The infinite density layer ensures that momentum is conserved on each side. So if observers inside of the cloak wished to reconstruct incoming waves (i.e., hear), they would have to be able to detect fields right at this discontinuity (or able to detect fields infinitesimally outside the inner lining, which will likely be the case for real implementations with finite thicknesses and nonsingular materials). And even then, they could only hear a few components.

There is no ρ^{-1} in Eq. (30), however, so this argument cannot be used to rule out the transmission of a nonzero pressure wave by the cloak. Considering Eq. (25) when $kR=0$,

$$p = \frac{1}{\sqrt{4\pi}}p_{00}. \tag{33}$$

So in a perfect cloak, it is possible to partially reconstruct signals sent in from the outside while still having the presence of the cloak be completely undetected.

This would appear to contradict Ruan *et al.* [24], Chen *et al.* [25], and Cummer *et al.* [17] who all found the field inside of a cloak to be strictly 0. A more careful reading shows that there is no such contradiction. Ruan *et al.* dealt with a cylindrical system. While this system is analogous to the spherical one considered here, there are differences. In particular, there were singularities in both the radial and angular μ matrix elements (ρ here). Their argument relied specifically on the divergence of μ_θ at the inner lining.

Chen *et al.* did specialize to the same material parameters in a spherical geometry (which is no surprise, as we explicitly use their construction of the cloak). But because they focused upon electrostatics they only dealt with the potential (by analogy p). So the behavior of $\vec{P}$ is left unconstrained there. Furthermore, when considering the scattering solution to the field in each region, they neglected the j_0 term, which is the precisely the only nontrivial term. A justification for neglecting this term may be found in Qiu *et al.* [26]. They considered a spherical geometry and electromagnetism, so their $\vec{E}$ and $\vec{H}$ fields contained a j_0 term. However, they preferred to solve everything using potentials, where the summation begins with j_1. So, in both these papers, it is the preference to focus upon the potential that inspires their exclusion of j_0. Since the equations governing p are exactly analogous to those of the potential [16], we expect that $p_{00}=0$ in most cases.

Cummer *et al.* did include the j_0 term and presented two arguments for why it does not contribute to the interior field. Their first argument is to claim an analogy with Ruan *et al.*, which does not account for the differences between the two cloaks. Their second argument is that the spherical Hankel function h_0 should be included in Eq. (27) to maintain a strictly 0 field inside the cloak. This argument sounds reasonable, but introduces complications. This term must be introduced in a way to maintain $A_l h_l \equiv 0$ outside the cloak, where A_n is the weighting coefficient. Otherwise, there will be noticeable scattering and the invisibility will be lost. Furthermore, when we find P_r, this will introduce an h_1 term. This must also be properly behaved at both linings of the cloak to keep from breaking the invisibility or introducing internal fields. Their simulations indicated that all of these scruples are apparently accounted for in practice.

For any attempt at a reconstruction of the incoming wave, the ability to detect the nonzero momentum at the lining would be more important than the ability to detect the pressure wave. First of all, the $l=0$ component of the wave corresponds to a completely uniform pressure over the entire surface of the sphere—and therefore would be indistinguishable from a change in gauge pressure. The maximum detectable components of the momentum wave (i.e., just the $l=1$ terms), on the other hand, have the same angular dependence as a dipole moment in electromagnetism or a p-orbital in Hydrogen. It therefore would give a distinct signal that could not be mistaken for an overall shift in the gauge. Moreover, there are three different polarizations possible—corresponding to $m=-1,0,1$. This means that a comparatively richer amount of information could be sent to a cloaked observer by relying upon momentum instead of pressure—which only has one component.

In conclusion, we have examined the linearized acoustic wave equations and found a second field subject to the transformation media interpretation. In considering the behavior of the momentum in an acoustic wave, we have derived a field fundamentally different from other instances of the transformation media interpretation—its governing equation is distinct from the wave equation. The behavior of the momentum and pressure fields is then examined in the case of cloaking devices, where we find that perfect cloaking is preserved for both fields.

We thank Tom Philbin and Ulf Leonhardt for their numerous suggestions and helpful discussions. This work was supported through Cornell University's Rawlings Presidential Research Scholars Program.

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